The mathematics of Ponzi schemes

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Abstract

A first order linear differential equation is used to describe the dynamics of an investment fund that promises more than it can deliver, also known as a Ponzi scheme. The model is based on a promised, unrealistic interest rate; on the actual, realized nominal interest rate; on the rate at which new deposits are accumulated and on the withdrawal rate. Conditions on these parameters are given for the fund to be solvent or to collapse. The model is fitted to data available on Charles Ponzi's 1920 eponymous scheme and illustrated with a philanthropic version of the scheme.

Key words: Ponzi scheme, Charles Ponzi, differential equation, investment

1. Introduction

On December 11, 2008, Bernard L. Madoff was arrested for allegedly running a multibillion-dollar Ponzi scheme in which investors across the world lost $ 50 billion dollars (Henriques and Kouwe, 2008). The principle of such a scheme is simple: you entice potential investors by promising a high rate of return, which you cannot possibly deliver. The only way you can pay the

promised interest is by attracting new investors whose money you use to pay interests to those already in the fund.

Various macroeconomic models of rational Ponzi games have been described in the literature (O'Connell and Zeldes, 1998; Bhattacharya, 2003; Blanchard and Weil, 2001; Forslid, 1998). Micro Ponzi schemes have proliferated particularly on the Internet to the point that regulators fear "Ponzi-monium" (Blas, 2009). Here we wish to investigate the mathematics of Ponzi schemes by going beyond the simplistic pyramid-type explanations that rely on a more or less rapid doubling of the number of new investors.

Madoff's fund is only the most recent and perhaps biggest Ponzi scheme in history. Madoff offered a supposedly safe 10 % return, which was considered unrealistic, particularly in the financial climate prevailing at the end of 2008. Still, if he could invest the money at a two or three percent interest rate and if new deposits came in fast enough he could use the fresh money to pay those who withdrew their earnings. But how fast must new deposits come in? How long can the scheme last? What parameters drive the dynamics of the fund?

In this paper we provide some answers to these questions with a simple model that captures the main features of a Ponzi scheme. The model is described in section 2 with detailed results on the behavior of the fund as a function of seven parameters. In section 3 we describe Charles Ponzi's eponymous 1920 scheme and crudely fit the model to the data available (see Zuckoff (2005) and Dunn (2004) for biographies of this colorful character). We also discuss an improbable "philanthropic Ponzi scheme" that may have something to do with Social Security. Highlights and the main results are

brought together in the concluding section 4.

2. The model

2.1. Assumptions

We assume that the fund starts at time $t = 0$ with an initial deposit $K \geq 0$ followed by a continuous cash inflow $s(t)$. We next assume a promised rate of return r_p and a nominal interest rate r_n at which the money is actually invested. If $r_n \geq r_p$ then the fund is legal and has a profit rate $r_n - r_p$. If $r_n < r_p$ the fund is promising more than it can deliver. The promised rate r_p may be called the "Ponzi rate" and is equal to 0.10 in the introductory example. The nominal interest rate r_n is 0.02 or 0.03 in that example.

We need to model the fact that investors withdraw at least some of their money along the way. The simplest way of doing this is to assume a constant withdrawal rate r_w applied at every time t to the promised accumulated capital. The withdrawal at time t by those who invested the initial amount K is $r_w K e^{t(r_p - r_w)}$. If r_w is less than the promised rate r_p then these withdrawals increase exponentially; r_w can also be larger than r_p in which case withdrawals decrease exponentially as these investors are eating into the capital K.

In order to calculate the withdrawals at time t from those who added to the fund between times 0 and t we note that those who invested $s(u)$ at time u will want to withdraw at time $t > u$ a quantity $r_w s(u) e^{(r_p - r_w)(t-u)}$. Integrating these withdrawals from 0 to t and adding the previously calculated withdrawals from the initial deposit K yields the total withdrawals at time t

$$W(t) \overset{\text{def.}}{=} r_w \left(K e^{t(r_p - r_w)} + \int_0^t s(u) e^{(r_p - r_w)(t-u)} du \right). \qquad (1)$$

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We note that the nominal interest rate r_n does not appear in $W(t)$: withdrawals are based only on the promised rate of return r_p.

2.2. The differential equation

If $S(t)$ is the amount in the fund at time t then $S(t + dt)$ is obtained by adding to $S(t)$ the nominal interest $r_n S(t)dt$, the inflow of fresh money $s(t)dt$ and subtracting the withdrawals $W(t)dt$:

$$S(t + dt) = S(t) + dt[r_n S_a(t) + s(t) - W(t)]. \tag{2}$$

For $dt \to 0$ the amount $S(t)$ is the solution to the first order linear differential equation

$$\frac{S(t)}{dt} = r_n S_a(t) + s(t) - W(t). \tag{3}$$

We let $C = S(0)$ be the initial condition which may or may not be equal to K, the initial deposit made by customers. The fund managers can make an initial "in-house" deposit $K_0 \geq 0$, which will also be invested at the nominal rate r_n. In this case the initial value $C = K_0 + K$ is larger than K. An initial condition $C < K$ formally corresponds to the case where for some reason (theft or other) a fraction of the initial deposits K is not available. We will see later that the solution to the differential equation with an initial condition $C = S(0)$ other than K will be used when there is at some subsequent time t^* a sudden change in parameter values. (For example the cash inflow or withdrawal rate changes at t^*).

A simple assumption one can make on the cash inflow $s(t)$ is that of exponential growth:

$$s(t) = s_0 e^{r_i t} \tag{4}$$

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where s_0 is the initial density of the deposits and r_i will be called the investment rate. The withdrawals function $W(t)$ of Eq. (1) is now

$$W(t) = r_w e^{t(r_p - r_w)} \left(K + s_0 \frac{e^{t(r_w + r_i - r_p)} - 1}{r_w + r_i - r_p} \right) \tag{5}$$

where the fraction should be taken equal to t when $r_w + r_i - r_p = 0$.

The solution $S(t)$ to the differential equation (3) has a closed-form expression which will be formulated using the function

$$g(t, a, b, c, d, \alpha) \stackrel{def.}{=} ae^{bt} + ce^{dt} + \alpha. \tag{6}$$

With this notation $S(t)$ is

$$S(t) = g(t, a, b, c, d, \alpha)e^{r_n t} = ae^{(b+r_n)t} + ce^{(d+r_n)t} + \alpha e^{r_n t}. \tag{7}$$

where

$$a \stackrel{def.}{=} \frac{r_w[s_0 - (r_i - r_p + r_w)K]}{(r_p - r_n - r_w)(r_i - r_p + r_w)}, \tag{8}$$

$$b \stackrel{def.}{=} r_p - r_n - r_w, \tag{9}$$

$$c \stackrel{def.}{=} \frac{s_0(r_i - r_p)}{(r_i - r_n)(r_i - r_p + r_w)}, \tag{10}$$

$$d \stackrel{def.}{=} r_i - r_n, \tag{11}$$

$$\alpha \stackrel{def.}{=} C - \frac{s_0(r_n - r_p) + Kr_w(r_i - r_n)}{(r_i - r_n)(r_n - r_p + r_w)}. \tag{12}$$

The solution $S(t)$ of Eq. (7) is a linear combination of three exponentials, which we were not able to tackle directly by elementary methods. Indeed, the zeros of $S(t)$ and of its derivative can be calculated numerically but no closed forms were found, which precludes analytical results on the behavior of the function.

If however we know the number of positive zeros of $S(t)$ we can shed light on the conditions under which the fund is solvent ($S(t)$ remains positive). We will see that depending on parameter values $S(t)$ of Eq. (7) has 0, 1 or 2 positive zeros. When there is no positive zero then $S(t)$ remains positive and the fund is solvent. One positive zero means that $S(t)$ becomes negative and the fund has collapsed. Two positive zeros mean that $S(t)$ becomes negative, reaches a negative minimum, then becomes positive again. The fund has collapsed but could recover with a bailout equal to the absolute value of the negative minimum. (See Bhattacharya (2003) for an economist's bailout model of a Ponzi scheme). To simplify we will say in this case that the fund has collapsed then recovered.

Analytical results on the number of positive zeros will be obtained by noting that the zeros of $S(t)$ are also those of $g(t, a, b, c, d, \alpha)$ of Eq. (6). This function is a linear combination of only two exponentials plus a constant. The zeros still cannot be found in closed form by elementary methods. However the derivative of $g(t, a, b, c, d, \alpha)$ is a linear combination of two exponentials with no constant, which can be studied analytically. The following proposition provides results on the number of positive zeros of $g(t, a, b, c, d, \alpha)$.

Proposition 1. *We consider the function $g(t, a, b, c, d, \alpha)$ of Eq. (6) in the non-trivial case $a, b, c, d \neq 0$ and $b \neq d$. We also assume that $g(0, a, b, c, d, \alpha) \geq 0$. We consider the following set of two conditions:*

$$U \stackrel{def.}{=} \frac{cd}{ab} < 0, \quad V \stackrel{def.}{=} \frac{1 + cd/ab}{b - d} < 0. \tag{13}$$

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The function $g(t, a, b, c, d, \alpha)$ has an extremum

$$m \stackrel{def.}{=} a \left(\frac{-cd}{ab} \right)^{\frac{b}{b-d}} + c \left(\frac{-cd}{ab} \right)^{\frac{d}{b-d}} + \alpha \tag{14}$$

at the positive value

$$t_c \stackrel{def.}{=} \frac{\ln \left(\frac{-cd}{ab} \right)}{b-d} \tag{15}$$

if and only if Condition (13) is satisfied.

We have the following results, broken down into fours cases A_1, A_2, A_3, A_4, on the number of positive zeros of $g(t, a, b, c, d, \alpha)$:

A_1: Condition (13) is satisfied and $ab + cd < 0$. If $m > 0$ then $g(t, a, b, c, d, \alpha)$ has no positive zero (and therefore remains positive). If $m < 0$ and $b, d, \alpha < 0$ then the function $g(t, a, b, c, d, \alpha)$ has exactly one positive zero at a value smaller than t_c. In all other cases with $m < 0$ the function has one positive zero on each side of t_c.

A_2: Condition (13) is satisfied and $ab + cd > 0$. If $b, d < 0$, $\alpha > 0$ then the function has no positive zero. In all other cases there is one positive zero.

A_3: Condition (13) is not satisfied and $ab + cd < 0$. If $b, d < 0$, $\alpha > 0$, the function has no positive zero. In all other cases there is one positive zero.

A_4: Condition (13) is not satisfied and $ab + cd > 0$. There is no positive zero.

Proof. The proof is elementary and hinges on the following facts:

1. The derivative $g'(t, a, b, c, d, \alpha)$ equals 0 at t_c and m is the value of $g(t, a, b, c, d, \alpha)$ at t_c; t_c is positive if and only if Condition (13) is satisfied. The derivative $g'(0, a, b, c, d, \alpha)$ at 0 is equal to $ab + cd$.

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2. If b and d are negative the function $g(t, a, b, c, d, \alpha)$ tends to α for $t \to \infty$. If b or d is positive the function tends to $\pm\infty$ (depending on the signs of a, c).

3. If Condition (13) is not satisfied then either $g(t, a, b, c, d, \alpha)$ has an extremum for a negative value of t or no extremum at all. In both cases the function $g(t, a, b, c, d, \alpha)$ for $t > 0$ is monotone increasing if $ab + cd > 0$ and monotone decreasing otherwise.

$\square$

The limiting cases $ab + cd = 0$, $m = 0$, etc. pose no difficulty and are left as exercises.

2.3. Main result

In order to apply Proposition 1 to the parameters a, b, c, d, α of Eqs. (8)-(12) we first define

$$\rho \stackrel{def.}{=} r_i - r_p, \quad \sigma_K \stackrel{def.}{=} \frac{K r_w}{s_0} - 1. \tag{16}$$

We will need the function

$$C_1(K) \stackrel{def.}{=} \frac{s_0(r_n - r_p) + K r_w(r_i - r_n)}{(r_i - r_n)(r_n - r_p + r_w)} \tag{17}$$

which is the critical value of C above which α of Eq. (12) is positive.

We define the function

$$Z(K) \stackrel{def.}{=} \frac{\frac{K}{s_0}(r_w + r_i - r_p) - 1}{(r_i - r_p)/r_w}, \tag{18}$$

and note that $Z(K) = 1$ if and only if $K = s_0/r_w$ (i.e. $\sigma_K = 0$).

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The extremum m of Eq. (14) and the corresponding t_c of Eq. (15) are

$$m = s_0 \frac{(r_p - r_i)Z(K)^{\frac{r_n - r_i}{r_p - r_i - r_w}}}{(r_i - r_n)(r_n - r_p + r_w)} + C - C_1(K), \qquad (19)$$

$$t_c = \frac{\ln(Z(K))}{r_w + r_i - r_p}. \qquad (20)$$

We also define the function $C_2(K)$ as

$$C_2(K) \overset{def.}{=} C_1(K) + s_0 \frac{(r_p - r_i)Z(K)^{\frac{r_n - r_i}{r_p - r_i - r_w}}}{(r_i - r_n)(r_n - r_p + r_w)} \text{ if } K \geq s_0/r_w, \quad (21)$$

$$C_2(K) \overset{def.}{=} 0 \text{ if } K < s_0/r_w. \qquad (22)$$

The quantity $C_2(K)$ of Eq. (21) is the critical value of C above which the extremum m of Eq. (19) is positive.

With these notations we have the following result on the number of positive zeros of $S(t)$ of Eq. (7).

Theorem 1. *We consider the solution $S(t)$ of Eq. (7) defined by the non-negative parameters K, C, s_0, r_i, r_w, r_p and r_n. The number of positive zeros of $S(t)$ is given as a function of the sign of ρ (Figure 1):*

Case B_1: $\rho > 0, (r_i > r_p)$.

Sub-case $B_{1,1}$: $\sigma_K < 0$ ($K < s_0/r_w$). $S(t)$ has no positive zero.

Sub-case $B_{1,2}$: $\sigma_K > 0$ ($K > s_0/r_w$). We first consider the case $r_n > r_i$. If $C > C_2(K)$ (which includes the case $C = K$) then $S(t)$ has no positive zero for $t > 0$ and therefore remains positive for all $t > 0$. For $C_1(K) < C < C_2(K)$ the function $S(t)$ has one positive zero on each side of t_c. For $C < C_1(K)$ the function $S(t)$ has one positive zero. When $r_p < r_n < r_i$ the

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function $S(t)$ has one positive zero for $C < C_2(K)$ (which includes the case $C = K$) and none if $C > C_2(K)$. When $r_n < r_p$ the function $S(t)$ has one positive zero for $C < C_2(K)$ (which includes the case $C = K$ if K is larger than the fixed point $K^* = C_2(K^*)$ of $C_2(K)$) and none if $C > C_2(K)$ (which includes the case $C = K$ if K is smaller than the fixed point K^*).

Case B_2: $\rho < 0, (r_i < r_p)$.

Sub-case $B_{2,1}$: $r_w < r_p - r_n$ or $r_n < r_i$. The function $S(t)$ has one positive zero.

Sub-case $B_{2,2}$: $r_w > r_p - r_n$ and $r_n > r_i$. For $C > C_1(K)$ (which includes the case $C = K$ if $r_n > r_p$) the function $S(t)$ has no positive zero. For $C < C_1(K)$ (which includes the case $C = K$ if $r_n < r_p$) then $S(t)$ has one positive zero.

Proof. The application of Proposition 1 hinges on the following observations:

1. The parameter α of Eq. (12) is positive if and only if $C > C_1(K)$.

2. The extremum m of Eq. (14) is positive if and only if $C > C_2(K)$.

3. The difference $C_2(K) - C_1(K)$ has the same sign as $(r_p - r_i)/[(r_i - r_n)(r_n - r_p + r_w)]$.

4. For $\rho > 0$ the function $C_2(K)$ is an non decreasing function of $K > 0$ that has no positive fixed point if $r_n > r_p$ and one positive fixed point $K^* = C_2(K^*)$ if $r_n < r_p$.

5. The parameter σ_K and the derivative

$$g'(0, a, b, c, d, \alpha) = ab + cd = s_0 - Kr_w = -s_0\sigma_K \qquad (23)$$

of $g(t, a, b, c, d, \alpha)$ at 0 have opposite signs.

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With a, b, c, d, α of Eqs. (8)-(12) the quantities U and V of (13) are

$$U = \frac{\rho}{-r_w \sigma_K - (\sigma_K + 1)\rho}, \quad V = \frac{\sigma_K}{-r_w \sigma_K - (\sigma_K + 1)\rho}, \quad (24)$$

and are both negative if and only if ρ and σ_K have the same sign (because $\sigma_K + 1 > 0$).

In the case B_1 ($\rho > 0$) we consider two sub-cases.

Sub-case $B_{1,1}$: $\sigma_K < 0$. Condition (13) is not satisfied and the derivative of $g(t, a, b, c, d, \alpha)$ at 0 is positive. The result follows from A_4 of Proposition 1.

Sub-case $B_{1,2}$: $\sigma_K > 0$. Condition (13) is satisfied and the derivative of $g(t, a, b, c, d, \alpha)$ at 0 is negative. The results follow from A_1 of Proposition 1.

In both sub-cases of B_2 ($\rho < 0$) the proof relies on the sign of σ_K. When $\sigma_K < 0$ Condition (13) is satisfied and the derivative of $g(t, a, b, c, d, \alpha)$ at 0 is positive. The sub-cases $B_{2,1}$ and $B_{2,2}$ correspond to $b = r_p - r_n - r_w$ or $d = r_i - r_n$ positive and to b and d negative, respectively. The results follow from A_2 of Proposition 1. When $\sigma_K > 0$ Condition (13) is not satisfied and the derivative of $g(t, a, b, c, d, \alpha)$ at 0 is negative. The results follow from A_3 of Proposition 1. $\square$

2.4. Interpretation of results

Theorem 1 breaks down the results depending on whether the rate r_i of new investments is larger or smaller than the promised rate of return r_p.

We first consider the case when r_i is larger than r_p (Case B_1) and the nominal rate of return r_n is also larger than r_p (legal fund). In the sub-case $B_{1,1}$ ($K < s_0/r_w$) the fund is solvent ($Z = 0$) regardless of the initial
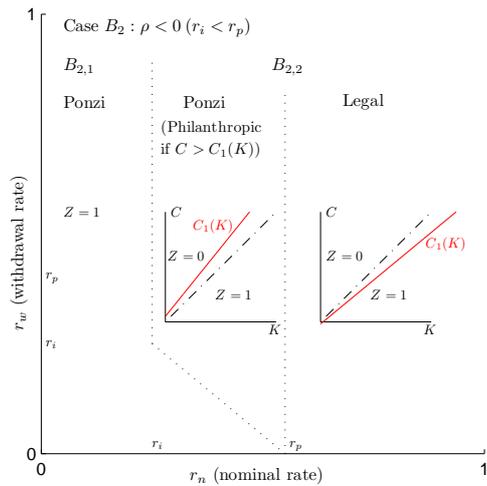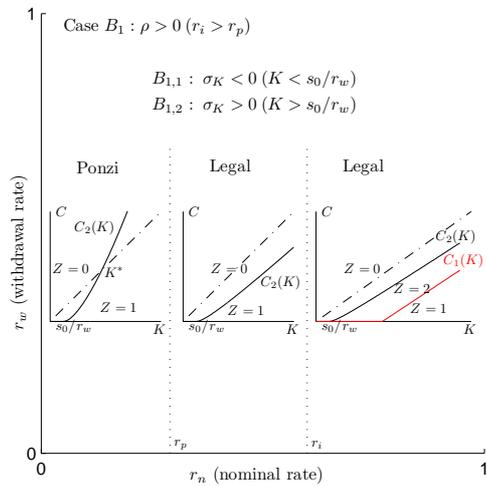
Figure 1: Number Z of zeros of the function $S(t)$ in the cases B_1 ($\rho > 0$) and B_2 ($\rho < 0$); Z is given in the phase space (r_n, r_w). In the sub-case $B_{2,1}$ the number Z of zeros is 1 regardless of the values of C and K. In the two sub-cases of $B_{2,2}$ and in the three regions of B_1, the number Z is given in the phase space (K, C).

condition C. In the sub-case $B_{1,2}$ $(K > s_0/r_w)$ the two rightmost graphs in Figure 1 (Case B_1) show that the fund remains solvent when C remains above $C_2(K)$ (which includes the case $K = C$ because the first diagonal (dash-dot line) is above $C_2(K)$). For r_n between r_p and r_i the fund collapses $(Z = 1)$ as soon as C drops below $C_2(K)$ (second graph). For r_n larger than r_i the fund collapses but recovers $(Z = 2)$ if C does not fall too much below $C_2(K)$ $(C_1(K) < C < C_2(K)$, third graph). If C is too small $(C < C_1(K))$ then the fund collapses $(Z = 1)$.

The first graph of Figure 1, Case B_1, shows what happens in a Ponzi scheme $(r_n < r_p)$ even with a rate of new investments r_i larger than r_p. The fund will be solvent for $C = K$ only if K is not too large (K less than the fixed point K^*). If $C = K$ and is larger than the fixed point K^*, then the combined withdrawals by the initial and subsequent investors eventually cause the fund to collapse.

In Case B_2 (the rate of new investments r_i is smaller than the promised rate of return r_p) we first consider the sub-case of $B_{2,2}$ with $r_n > r_p$ (legal fund, second graph). The fund remains solvent for $C > C_1(K)$, which included the case $C = K$. In the Ponzi sub-case of $B_{2,2}$ with $r_i < r_n < r_p$ and $r_w > r_p - r_n$ the fund does not grow too fast and is solvent if C is larger than $C_1(K)$, which is itself larger than K (first graph). This means that despite an r_i and an r_n smaller than r_p, the Ponzi scheme is solvent if the fund manager can add to K an "in-house" investment K_0 at least equal to $C_1(K) - K$. We will see in the numerical illustrations that $C_1(K) - K$ can be quite large and the scheme unprofitable for the fund manager, hence the "philanthropic Ponzi scheme" characterization. We emphasize for future reference that this

scenario hinges on an investment rate r_i that remains smaller than the nominal rate r_n. If the manager does not invest enough ($C < C_1(K)$) the fund collapses.

The Ponzi sub-case $B_{2,1}$ consists of the values $r_n < r_i$ and of the values (r_n, r_w) for which $r_w < r_p - r_n$ and r_n is between r_i and r_p (triangular region below the first graph). In this sub-case $B_{2,1}$ the fund grows too fast and collapses ($Z = 1$).

This analysis shows that the role of r_w is ambiguous when r_n is between r_i and r_p. Although a small r_w ($r_w < r_p - r_n$, $B_{2,1}$) may seem desirable, the fund will grow more in the long run and eventually collapses. A large r_w ($r_w > r_p - r_n$, $B_{2,2}$) may seem dangerous but depletes the fund and means smaller withdrawals in the long run. The fund is ultimately solvent if C is large enough to absorb the large early withdrawals ("philanthropic Ponzi scheme").

2.5. Actual and promised amounts in the fund

In order to describe the dynamics of a fund that includes a sudden parameter change at some time t^*, we need to make explicit the role of the parameters by denoting $S(t, K, C, s_0, r_i, r_w, r_p, r_n)$ the solution of the differential equation given in (7).

We introduce the actual and promised amounts $S_a(t)$ and $S_p(t)$. The actual amount $S_a(t)$ in the fund (based on the nominal rate of return r_n and the initial condition C), is the one given in Eq. (7) and rewritten explicitly as

$$S_a(t) = S(t, K, C, s_0, r_i, r_w, r_p, r_n) = \frac{r_w[s_0 - (r_i - r_p + r_w)K]}{(r_p - r_n - r_w)(r_i - r_p + r_w)} e^{(r_p - r_w)t} +$$

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$$\frac{s_0(r_i - r_p)}{(r_i - r_n)(r_i - r_p + r_w)} e^{r_i t} + \left(C - \frac{s_0(r_n - r_p) + Kr_w(r_i - r_n)}{(r_i - r_n)(r_n - r_p + r_w)} \right) e^{r_n t} \quad (25)$$

The quantity $S_p(t)$ is the amount that is promised to and belongs to investors; $S_p(t)$ is obtained by setting in Eq. (25) the parameter r_n equal to r_p and the initial condition C equal to K. Under these conditions the third term in Eq. (25) becomes zero and

$$S_p(t) = S(t, K, K, s_0, r_i, r_w, r_p, r_p) =$$

$$\frac{s_0}{r_p - r_i - r_w} \left(e^{(r_p - r_w)t} - e^{r_i t} \right) + Ke^{(r_p - r_w)t} \quad (26)$$

Contrary to the actual amount $S_a(t)$ in the fund the promised amount $S_p(t)$ is positive regardless of the parameter values.

2.6. Change in parameter values

The assumption of an exponentially increasing density of new investments is not realistic in the long run and we may wish to examine what happens if the investment rate r_i suddenly drops to 0. This means that the flow of new investments becomes a constant. More generally, it would be useful to be able to describe the future dynamics of the fund if at a point in time t^* the parameters $(s_0, r_i, r_w, r_p, r_n)$ experience a sudden (discontinuous) change of value and become $(s_0', r_i', r_w', r_p', r_n')$.

We call C' and K' the actual and the promised amounts in the fund at time t^*:

$$C' = S_a(t^*) = S(t^*, K, C, s_0, r_i, r_w, r_p, r_n) \quad (27)$$

and

$$K' = S_p(t^*) = S(t^*, K, K, s_0, r_i, r_w, r_p, r_p). \quad (28)$$

These quantities will be the new initial condition and initial investment starting at time t^*. The actual and promised amounts at any time t are now

$$S_a(t) = \begin{cases} S(t, K, C, s_0, r_i, r_w, r_p, r_n) & \text{if} \quad t \le t^*; \\ S(t - t^*, K', C', s_0', r_i', r_w', r_p', r_n') & \text{if} \quad t > t^*. \end{cases} \tag{29}$$

and

$$S_p(t) = \begin{cases} S(t, K, K, s_0, r_i, r_w, r_p, r_p) & \text{if} \quad t \le t^*; \\ S(t - t^*, K', K', s_0', r_i', r_w', r_p', r_p') & \text{if} \quad t > t^*. \end{cases} \tag{30}$$

Several discontinuous parameter changes at different times can be dealt with in this fashion.

3. Applications

3.1. Charles Ponzi's original scheme

Ponzi schemes are named after the eponymous Italian adventurer and fraudster Charles Ponzi (1882-1949) who emigrated to the United States in 1903. We will use the data available on Ponzi's scheme to fit crudely the model and study the implications of a density of new deposits becoming constant ($r_i = 0$) at some point t^*. The data used comes from DeWitt (2009).

Charles Ponzi was a petty crook who spent time in and out of prison. While in Boston in 1919 he learnt that International Reply Coupons could be bought in one country and theoretically resold in another at a profit. Trading in these coupons was not realistic, but that did not prevent Ponzi from issuing bonds which offered a 100% profit if held for 90 days (0.25 years). We translate this 90-day doubling time into an annualized rate of return r_p that must satisfy $e^{0.25 r_p} = 2$, i.e. $r_p = 2.773$.
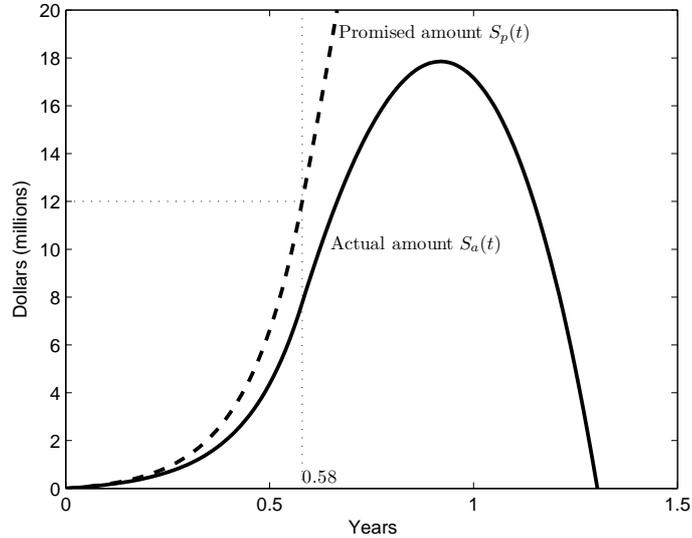
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Figure 2: Actual and promised amounts $S_a(t)$ and $S_p(t)$ in Ponzi's fund, crudely fitted to available data for the duration of the fund from time $t = 0$ (December 26, 1919) to $t^* = 0.58$ (July 26, 1920). The hypothetical trajectories beyond 0.58 illustrate the effect of the flow of new investments becoming constant and equal to the $ 200,000 a day reached at time 0.58. The fund would have collapsed about nine months later ($S_a(1.30) = 0$).

Ponzi was penniless when he launched his fund and there is no reported initial investment K, so we can safely set $C = K = 0$. The parameters s_0 and r_i for the density of new investments $s_0 \exp(r_i t)$ can be estimated crudely on the basis of information on deposits made between the first day (December 26, 1919, $t = 0$) and last day (July 26, 1920, $t^* = 0.58$) of the scheme's history. On the last day Ponzi collected $200,000 for a total of $10 million deposited over the seven-month period. Under the assumption of exponential growth and with the year and a million dollars as the time and monetary

units, the parameters s_0 and r_i must satisfy

$$s_0 e^{0.58r_i} = 0.2 \times 365, \ \ s_0 \left(\frac{e^{0.58r_i} - 1}{r_i} \right) = 10 \tag{31}$$

which yields

$$s_0 = 1.130, \ r_i = 7.187. \tag{32}$$

This value of s_0 translates into an initial flow $s_0/365$ of \$ 3,095 a day. An exponential growth from \$ 3,095 to \$ 200,000 a day seems plausible in view of Ponzi's own description of *A huge line of investors, four abreast, stretched from the City Hall Annex . . . all the way to my office!* (quote in DeWitt (2009)).

The actual profit made on the International Reply Coupons was negligible. We thus arbitrarily choose a nominal rate r_n of 0.01, although r_n could just as well be set equal to 0. Indeed, the impact of a one or even five percent nominal return is negligible in comparison with new investments pouring in at an instantaneous rate of 7.187.

The greatest uncertainty rests with the redemption rate (our withdrawal rate r_w) at which investors cashed in the bonds. We only know that some investors redeemed their bonds after 90 days but that many left their money to double once again. We can however estimate r_w crudely on the basis of information available at the time of Ponzi's trial. He redeemed \$5 million of his bonds after the July 1920 cessation of activity with \$7 million still outstanding. We will take these figures to mean that the promised amount accumulated by time $t^* = 0.58$ is

$$S_p(0.58) = 12. \tag{33}$$

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	K	C	s_0	r_i	r_w	r_p	r_n
Ponzi's scheme ($t \leq 0.58$)	0	0	1.130	7.187	1.47	2.773	0.01
Ponzi's scheme ($t > 0.58$)	12	7.76	73	0	1.47	2.773	0.01
Philanthropic Ponzi scheme	0	280	1	0	0.12	0.15	0.04

Table 1: Model parameter values for Charles Ponzi's 1920 scheme (Section 3.1) and for a hypothetical philanthropic Ponzi scheme (Section 3.2).

With all other parameter values known, we solved this equation numerically for r_w and obtained $r_w = 1.47$ (Table 1). The corresponding actual amount in the fund at $t^* = 0.58$ is then $S_a(0.58) = \$7.763$ million. After the late \$5 million redemption Ponzi was left with \$2.763 million, which is consistent with his reported assets of \$2 million at the time of his trial if we assume he had spent \$0.763 million.

Figure 1 shows that with $r_n < r_p < r_i$ and $K = C = 0$ the scheme corresponds to the first graph of Case B_1 and to the sub-case $B_{1,1}$. The function $S_a(t)$ has no zero and would continue to grow at the asymptotically exponential rate r_i (second term of Eq. (25)) as long as investors pour money in at the same rate r_i.

Figure 2 depicts the actual and promised amounts up to $t^* = 0.58$ on the basis of the parameter values estimated above. As a projection exercise we assume that after $t^* = 0.58$ the flow of new deposits stopped increasing and remained equal to the \$200,000 per day reached at $t^* = 0.58$ (July 26, 1920). This means that the annualized density is now $s_0' = 200,000 \times 365 = \$73$ million a year and that r_i became $r_i' = 0$. The other parameter values were kept unchanged after $t^* = 0.58$ ($r_n' = 0.01$, $r_p' = 2.773$, $r_w' = 1.47$, Table

1). With $r'_i = 0 < r'_n = 0.01 < r'_p = 2.773$ we are now in the sub-case $B_{2,1}$ of B_2 because $r'_w = 1.47 < r'_p - r'_n = 2.763$. As expected the fund would have collapsed, with $S_a(t)$ reaching 0 roughly nine months after t^* $(S_a(1.30) = 0)$. The promised amount $S_p(t)$ continues to grow exponentially at the rate $r'_p - r'_w = 1.303$.

3.2. A philanthropic Ponzi scheme

We noted that if $r_w > r_p - r_n$ and $r_i < r_n < r_p$, then for $C > C_1(K)$ the Ponzi scheme is solvent. We will now see why we labeled this scenario "philanthropic". With $C - C_1(K)$ positive, the asymptotic growth rate of the actual amount $S_a(t)$ is r_n because the last term $(C - C_1(K))e^{r_n t}$ in Eq. (25) dominates all other exponential terms. The promised amount $S_p(t)$ of Eq. (26) grows at the asymptotic rate $r_p - r_w$ which is smaller than the nominal r_n. The fund's profit is $S_a(t) - S_p(t)$ and we have for large t

$$S_a(t) \sim S_a(t) - S_p(t) \sim (C - C_1(K))e^{r_n t} \tag{34}$$

where $x(t) \sim y(t)$ means that $\lim\limits_{t\to\infty} x(t)/y(t) = 1$. If the initial deposit is $K = 0$ then the initial condition C is equal to the "in-house" investment K_0 and (34) becomes

$$S_a(t) \sim S_a(t) - S_p(t) \sim (C - C_1(0))e^{r_n t} = \left(C - \frac{s_0(r_p - r_n)}{(r_n - r_i)(r_n - r_p + r_w)}\right)e^{r_n t} \tag{35}$$

where C must be larger than $C_1(0)$. The actual amount $S_a(t)$ and the profit $S_a(t) - S_p(t)$ grow asymptotically at the exponential rate r_n but are both smaller than the profit $Ce^{r_n t}$ that the manager could have made by simply investing the quantity C at the nominal rate r_n. This shows that solvency
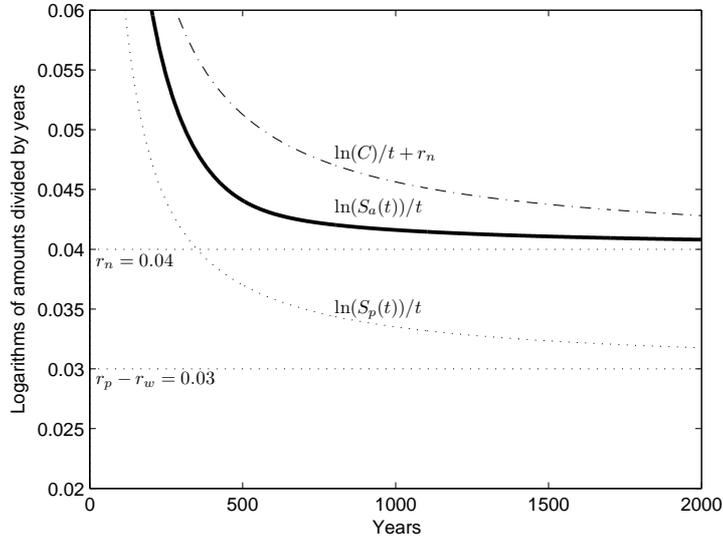
Figure 3: Natural logarithms divided by time t of actual and promised amounts in philanthropic Ponzi fund, compared with the larger natural logarithm divided by t of the accumulated amount had the \$ 280 million been invested at the nominal rate r_n $(\ln(C)/t+r_n)$.

hinges on a philanthropic fund manager who is willing to invest a significant initial amount and give away a share of her profits.

We will see how large the initial investment C must be with the example of a manager offering a 15% return to individuals who contribute one million dollars a year and continuously withdraw 12% of their accumulated capital $(r_p = 0.15, s_0 = 1, r_i = 0, r_w = 0.12$, Table 1). We assume the manager can realistically earn 4 % $(r_n = 0.04)$. With no initial deposits by investors $(K = 0)$ the fund manager's minimum investment C to keep the fund solvent is $C_1(0) = 275$. With $C = 280$ we plotted in Figure 3 the functions $\ln(S_a(t))/t$ and $\ln(S_p(t))/t$ to show their convergence to $r_n = 0.04$ and $r_p - r_w = 0.03$

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respectively. We also plotted $\ln(Ce^{r_n t})/t = \ln(C)/t + r_n$, which is the natural logarithm divided by time t of the accumulated amount had the \$280 million been invested at the nominal rate r_n. This quantity also converges to $r_n = 0.04$ while staying larger than $\ln(S_a(t))/t$ and $\ln(S_p(t))/t$.

An initial investment of \$280 million dollars to sustain (at a loss) a fund that grows by only \$1 million a year may not be a fund manager's idea of a profitable venture. This example was provided only to illustrate the fact that with a sufficiently large initial "in-house" investment a Ponzi scheme can be solvent with no or a very small growth in deposits and a nominal interest rate smaller than the promised one.

The example may also contribute to the discussion about the U.S. Social Security system being comparable to a Ponzi scheme. (See Mandel (2008) for a balanced account). The Social Security Administration vehemently denies any connection and suggests that the short-lived nature of Charles Ponzi's scheme is the telltale sign of such a fraudulent operation (DeWitt, 2009). The same Social Security source indicates that the German and U.S. Social Security systems have been in operation since 1889 and 1935, respectively. DeWitt (2009) implies that this longevity disqualifies these systems as Ponzi schemes. However our model makes clear that a Ponzi scheme can last a long time before collapsing or even indefinitely with or without a growth in deposits. In particular, our philanthropic Ponzi scheme which relies on a large initial investment may approximate a perfectly legal and legitimate non-profit government investment into a social security scheme meant to provide fixed incomes to a growing population of retirees. We recall however that in this case the solvency of the fund depends on an investment rate r_i

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that remains smaller than the nominal rate of return r_n. Although the role of population growth is beyond the scope of this paper, this shows that the retiree population and its investments must not grow too rapidly in order for the fund to be solvent.

4. Conclusion

After the collapse of Charles Ponzi's scheme in July 1920 it took seven years of litigation for the investors to get 37 cents on the dollar of their principal. After several trials Ponzi spent a total of 10 years in prison before returning to Italy where he briefly worked for Benito Mussolini. Having once again mismanaged things he fled to Brazil where he died a pauper in a Rio de Janeiro charity hospital in 1949.

Robbing Peter to pay Paul is an ancient practice that started long before Ponzi and is experiencing a comeback on the Internet. This paper attempts to add to our understanding of these schemes with the simplest possible continuous-time mathematical model of a fund that offers more than it can deliver. To be sure, constant investment, withdrawal and nominal interest rates capture only crudely the variety of human behaviors and the complexities of financial instruments available today. On the other hand, some of our assumptions are quite realistic. For example there is good reason to believe that setting r_n to 0 or 0.01 in the Ponzi or Madoff case is probably an accurate description of what they did with the money.

As we expected the fund is always solvent with $C = K$ in the case of a legal fund characterized by $r_n > r_p$. In a Ponzi scheme $(r_n < r_p)$ the fund can remain solvent depending on the values of the investment rate r_i and the

withdrawal rate r_w. The model sheds light on the ambiguous role played by these two parameters. If r_i is too large or r_w too small the fund grows fast and can be in jeopardy as withdrawals increase. If r_i is too small or r_w too large the fund may not be able to keep up with withdrawals.

Our model yields a variety of increasing trajectories that may look alike initially, but are fundamentally different in their long-run behavior. Some will continue to increase as long as new investments come in - others will increase possibly for a long time before they collapse. This happens when parameter values in the phase spaces of solutions are close to border regions between different qualitative behaviours (for example between no zero and one zero for the function $S(t)$). In some cases $S(t)$ initially decreases, reaches a positive or negative minimum, and then recovers.

Finally, our results can provide concrete answers to financial regulators and others confronted with funds that make unrealistic claims. Suppose one investigates a fund that promises a 10% return per month, provides its own initial "in house" investment of $5 millions dollars, claims a doubling of its deposits every two months and has investors who withdraw half their earnings every year. It is now possible to predict what will happen to the fund, at least under the simple assumptions made in the model described in this paper.

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